Non-isothermal flow of viscoelastic elastomer composites in a converging channel

A.V. Baranov
I.M. Gubkin Russian State University of Oil and Gas, Moscow

Highly filled polymer composites are very widely used in different processes of modern industry, and therefore the investigation of the flow of such media in different channels of processing equipment is being paid a great deal of attention. A mathematical model of the non-isothermal flow of a Newtonian fluid in a converging plane channel is presented. All the important factors that must be taken into account in the development of the model are noted. Many assumptions are made on the basis of the fact that flow occurs at low Reynolds numbers and high Peclet numbers. As the rheological model, the Maxwell higher convective model is used. The solution is sought using series, and all the stress components are expressed in the form of power series expansion. Thermal boundary conditions of the first kind, the temperature dependence of viscosity, and energy dissipation are taken into account. The problem of finding the temperature profiles is solved by an iteration scheme, at each step of which the collocation method is used. The results of calculations are given.

The flow of rubber mixes and different elastomer composites in a converging plane channel (Figure 1) is realised in flat-slit nozzles of extruders or in the gate connecting the sprue channel and the mould cavity. Also, this type of flow is encountered in processes of applying heat-resistant coatings to the surfaces of metal shells of articles of special designation, where a highly filled rubber-based composite is forced under high pressure through a flat-slit nozzle.

The process of flow of highly viscous rubber mixes through small-diameter channels is normally carried out at high temperature and pressure gradients with significant dissipative heat generation. Therefore, a mathematical model should include the temperature dependence of viscosity. Furthermore, it must be taken into account that elastomer composites are non-Newtonian media. In this context, we can refer primarily to studies [1, 2] where, on the basis of numerical methods, the non-isothermal flow of a power-law fluid in a slightly converging plane channel [1] and in a round tube [2] was considered. The temperature dependence of viscosity was adopted in exponential form, and the equation of energy included a dissipative function.

The flow in a converging channel is accompanied with considerable extension of the polymer in the direction of flow, which leads to accumulated elastic energy as the medium approaches the zone of discharge from the channel. In this type of flow, the high elasticity of polymers is clearly manifested. An appreciable role begins to be played by the first difference in normal stresses, which requires the introduction into the mathematical model of the rheological equation of the non-linear viscoelastic medium. In this connection, we can firstly cite studies [3–5] where the problem was solved for different rheological models with a temperature-independent viscosity. In a more developed formulation, taking into account the temperature dependence of viscosity, the
problem of non-isothermal flow of a viscoelastic fluid in a converging plane channel was solved by Baranov et al. [6, 7]. In the equation of energy, dissipative heat generation was taken into account. Here, it was assumed in the first of these studies by Baranov and Balinov [6] that near-wall slip can be ignored, while in the study by Baranov and Dakhin [7] the slip velocity at the wall was taken into account. From the rheological point of view, the medium is considered to be a non-linear viscoelastic fluid, the material constants of which are functions of temperature and the second invariant of the strain rate tensor. A second-order differential rheological model was selected, describing all characteristic features of the behaviour of polymer composites — abnormal viscosity and the first and second differences in normal stresses. According to this model, the stress tensor is determined by the kinematic tensors of strain rate $B_1$ and strain acceleration $B_2$:

$$
\tilde{\sigma} = -p \delta + \mu B_1 + \xi_1 B_2 + \xi_2 B_1^2 \quad (1)
$$

where $B_1$ and $B_2$ are termed the Rivlin–Eirichsen tensors. However, media described by differential models of this type are termed memoryless fluids, as the stress tensor components at a given moment in time do not take sufficient account of the prehistory of strains and stresses. In recent years, in the world literature, it has been considered a generally accepted fact that the best results when describing the flows of non-linear viscoelastic media are given by rheological models of the relaxation (velocity) type. One of these, relatively simple and effective, is Maxwell's higher convective model [8]:

$$
\tau + \lambda \left( \frac{\partial \tau}{\partial t} + \dot{\nu} \cdot \nabla \tau - \tau \cdot \nabla \dot{\nu} + \nabla \tau \cdot \tau \right) = \mu \left( \nabla \dot{\nu} + \nabla \dot{\nu}^T \right) \quad (2)
$$

where $\tau$ is the extra stress tensor, $\lambda$ is the relaxation time, $\dot{\nu}$ is the velocity vector, and $\mu$ is the viscosity at zero shear velocity.

It is well known [9] that for most elastomers at low deformation rates of the order of $10^{-2} - 10^{-4} \text{s}^{-1}$, the shear stress is proportional to the shear velocity. Here, the viscosity $\mu$ for basic rubbers lies in the range 1–10 MPa s.

Elastomer composites possess high viscosity, and therefore their flow occurs at low Reynolds numbers. This makes it possible firstly not to consider the hydrodynamic initial section, and to assume that the velocity profile at entry into the channel is developed. Secondly, inertia terms in the equation of motion can be ignored. It is also assumed that there are no circulation (secondary) flows in the convergent channel, i.e. we can confine ourselves simply to the single longitudinal velocity component $v_r = v_r(r, \varphi)$. In this case, from the discontinuity equation:

$$
\frac{\partial (rv_r)}{\partial r} + \frac{\partial v_r}{\partial \varphi} = 0 \quad (3)
$$

with $v_r = 0$, it follows that this velocity component can be presented in the form [4]:

$$
v_r = \frac{f(\varphi)}{r} \quad (4)
$$

where $f(\varphi)$ is a still unknown function.

Then, with account taken of the above, the equations of motion will take the form [4, 8]:

$$
\frac{\tau_{rr} - \tau_{\varphi\varphi}}{r} + \frac{\partial \tau_{r\varphi}}{\partial r} \frac{1}{r} \frac{\partial p}{\partial \varphi} = \frac{\partial p}{\partial r} \quad (5)
$$

$$
\frac{2}{r} \frac{\tau_{\varphi\varphi}}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} \frac{1}{r} \frac{\partial p}{\partial \varphi} = \frac{1}{r} \frac{\partial p}{\partial \varphi} \quad (6)
$$

where $\tau_{rr}, \tau_{r\varphi}$, and $\tau_{\varphi\varphi}$ are the stress tensor components, and $p$ is pressure.

Differentiating Equations (5) and (6) with respect to the variables $\varphi$ and $r$ respectively, pressure can be omitted. Then we will obtain the following equation:

$$
\frac{1}{r} \frac{\partial^2}{\partial \varphi^2} \left( r (\tau_{rr} - \tau_{\varphi\varphi}) \right) - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r^2 \tau_{r\varphi}) \right) + \frac{1}{r} \frac{\partial^2 \tau_{\varphi\varphi}}{\partial \varphi^2} = 0 \quad (7)
$$

Below, the following dimensionless variables are used:

$$
\sigma_{\varphi} = \frac{\lambda}{\mu} \tau_{\varphi}; \quad \overline{R} = \frac{r}{\lambda \overline{V}_r}; \quad R = \frac{r}{r_0}; \quad \overline{V}_r = \frac{V_r}{\overline{V}_r};
$$

$$
\pi = \frac{\lambda}{\mu} \rho; \quad F(\varphi) = \frac{f(\varphi)}{\lambda \overline{V}_r}; \quad \Phi = \frac{\varphi}{\alpha}
$$

where $\overline{V}_r$ is the average flow velocity at entry into the channel.

The distribution of the stress components is sought approximately in the form of an expansion into the following series [8, 10, 11]:

$$
\sigma_{rr}(R, \varphi) = \sum_{n=0}^{N} a_n(\varphi) \frac{1}{R^n}; \quad \sigma_{r\varphi}(R, \varphi) = \sum_{n=0}^{N} b_n(\varphi) \frac{1}{R^n};
$$

$$
\sigma_{\varphi\varphi}(R, \varphi) = \sum_{n=0}^{N} c_n(\varphi) \frac{1}{R^n} \quad (8)
$$

Substituting (8) into Equation (2), and equating the coefficients at identical powers $R^{-n}$, we can express the functions $a(\varphi), b(\varphi)$, and $c(\varphi)$ in terms of the function $F(\varphi)$ [8], confining ourselves in this case to the first six terms of series (8):
Having determined the coefficients \( a_n, b_n, \) and \( c_n \), it is then possible to substitute (8) into Equation (7) to find the unknown function \( F(\phi) \). Assembling terms with identical powers \( R^{-n} \), it is possible to obtain the following equation:

\[
4F' + F'''' = 0
\]  

(10)

Here, the boundary conditions have the form:

\[
\phi = \pm \alpha; \quad V_r = 0
\]  

(11)

\[
\phi = 0; \quad \frac{\partial V}{\partial \phi} = 0
\]  

(12)

The solution of Equation (9) under the prescribed boundary conditions is known [8]:

\[
F(\phi) = \frac{\cos(2\alpha) - \cos(2\phi)}{\sin(2\alpha) - 2\alpha \cos(2\alpha)}
\]  

(13)

Function (13) also satisfies the condition of constancy of flow rate:

\[
\int_{-1}^{1} V_r R d\Phi = -1
\]  

(14)

After determination of the function \( F(\phi) \), and with account taken of (9), it is possible to consider all stress fields [8] to be found. Then, integrating (5), we can find the pressure distribution along the channel. From the practical point of view, the pressure averaged over the cross-section of the channel \( \bar{p}(r) \) is of interest, and therefore the following equation is used to determine it:

\[
\frac{d\bar{p}}{dr} = \frac{1}{\alpha} \int_{0}^{a} \left( \tan \phi + \frac{\tan \phi}{r} + \frac{1}{r} \frac{d\tau_{rr}}{d\phi} \right) d\phi
\]  

(15)

The formulated problem is solved for the non-isothermal case where the viscosity is considered to be dependent on temperature:

\[
\mu = \mu_0 \exp\left[ b(T_w - T) \right]
\]  

(16)

where \( T \) is the temperature of the composite, \( T_w \) is the temperature of the channel walls, and \( b \) is the experimentally determined empirical constant.

For the non-isothermal case, the hydrodynamic problem must be solved at the same time as finding the temperature field. Rubber mixes possess low thermal conductivity. As a result, the flow of such media generally occurs at high Peclet numbers (\( Pe > 100 \)). This makes it possible, in the equation of energy, to ignore axial thermal conductivity by comparison with convective heat transfer. The equation of energy with account taken of energy dissipation has the form:

\[
\nu \frac{\partial^2 T}{\partial r^2} = \frac{1}{\rho c_p} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r} \frac{\partial \Psi}{\partial \phi} \right]
\]  

(17)

where \( \alpha \) is the thermal diffusivity, \( \rho \) is density, and \( c \) is the specific heat.

The dissipative function is determined in the following way:

\[
\Psi(r, \phi) = \tau_{rr} \frac{\partial \Psi}{\partial r} + \tau_{\phi\phi} \frac{\partial \Psi}{\partial \phi} + \frac{\tau_{r\phi}}{r} \frac{\partial \Psi}{\partial r} + \frac{\tau_{\phi\phi}}{r} \frac{\partial \Psi}{\partial \phi}
\]  

The thermal boundary conditions have the form:

\[
\phi = \pm \alpha; \quad T = T_w
\]  

(18)

\[
\phi = 0; \quad \frac{\partial T}{\partial \phi} = 0
\]  

(19)

\[
r = r_0; \quad T = T_0
\]  

(20)

The problem formulated is solved by an iteration scheme. At the first step, the hydrodynamic problem is solved for isothermal conditions, and the velocity fields and stress fields are found using (4), (8), (9), and (13). After this, the equation of energy (17) is solved. Here, at each iteration step, the temperature distribution in dimensionless form is sought in the following way:

\[
\theta = (1 - \Phi) \sum_{i=0}^{N} A_i(R) \Phi^i
\]  

(21)

where \( \theta = (T - T_w)/(T_0 - T_w) \).

The values of the functions \( A_i(R) \) at entry into the channel were determined using the boundary condition (20). The unknown functions \( A_i(R) \) were found by the collocation method, which was set out in detail by Baranov and Balinov [6]. After finding the temperature field, at the next iteration step a refined recalculation of the velocity, pressure, and stress distributions was carried out, with allowance for the temperature dependence of viscosity (16).

The mass-average temperature in dimensional and dimensionless form is defined in the following way:

\[
T_m = \frac{\int_{0}^{a} T V_r d\phi}{\int_{0}^{a} V_r r d\phi}; \quad \theta = \frac{\int_{0}^{a} \theta V_r d\phi}{\int_{0}^{a} V_r d\phi} = \bar{r} \int_{0}^{a} \theta V_r d\phi
\]  

(22)
On the basis of the obtained solution, the main parameters of the given process of flow were calculated. The values of material physical properties that are characteristic of moulding grades of rubber mixes were adopted. Figure 2 shows the distribution of the dimensionless pressure $P = \left( \rho_0 v_0^2 / \mu_r \right)$, and Figure 3 shows the change in the dimensionless mass-average temperature over the dimensionless length of the channel $R = r / r_0$.

**Figure 2.** The distribution of dimensionless pressure over dimensionless channel length

**Figure 3.** The change in dimensionless mass-average temperature over dimensionless channel length

REFERENCES