Combined Isotropic-Kinematic Hardening Laws with Anisotropic Back-stress Evolution for Orthotropic Fiber-Reinforced Composites

M.G. Lee, D. Kim, K. Chung*, J. R. Youn and T. J. Kang
School of Materials Science and Engineering, Seoul National University, 56-1 Shinlim-dong, Kwanak-gu, Seoul 151-742, Korea
*Research Institute of Advanced Materials, Seoul National University 56-1, Shinlim-dong, Kwanak-gu, Seoul, 151-742, Korea

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SUMMARY
In order to describe the Bauschinger and transient behavior of orthotropic fiber-reinforced composite solids, a combined isotropic-kinematic hardening law based on the non-linear kinematic hardening rule was considered here, in particular, based on the Chaboche type law. In this modified constitutive law, the anisotropic evolution of the back-stress was properly accounted for. Also, to represent the orthotropy of composite materials, Hill’s 1948 quadratic yield function and the orthotropic elasticity constitutive equations were utilized. Furthermore, the numerical formulation to update the stresses was also developed based on the incremental deformation theory for the boundary value problems. Numerical examples confirmed that the new law based on the anisotropic evolution of the back-stress complies well with the constitutive behavior of highly anisotropic materials such as fiber-reinforced composites.

INTRODUCTION
Recently, fiber-reinforced composites have been commonly used as high performance materials, which are subjected to various ranges of stresses, temperatures and loading conditions. As for their mechanical properties, fiber-reinforced composites are usually treated as linear elastic materials since fibers provide the majority of the strength and stiffness. However, more sophisticated description requires consideration of some form of plasticity, viscoelasticity, or both (viscoplasticity). Experimental studies under static loading confirm that fiber-reinforced composites show elasto-plastic behavior including the Bauschinger effect and transient behavior, hysteresis loops under cyclic loading and so on. Although significant progresses have been made over the years, little work has been done to implement those sophisticated material behaviors for structural applications. This article deals with the theory of plasticity for anisotropic fiber-reinforced composites, especially for orthotropic ones. The anisotropic yield function and the combined type of isotropic-kinematic hardening have been considered based on elasto-plasticity.

Previous efforts to implement plasticity considered three types of hardening assumptions, which are schematically illustrated in Figure 1: isotropic, kinematic and combined type hardening. For example, Leewood et al. used the isotropic hardening to describe plastic properties of fibrous metal composites. Isupov et al. dealt with the micromechanical analysis of plastic deformation processes of metal matrix composites composed of elastic reinforcing elements and the plastic matrix, based on the generalized anisotropic Prager kinematic hardening law. Recently, Sarbayev described the plasticity theory for anisotropic composites utilizing the kinematic hardening of the anisotropic quadratic yield function. However, few of the previous works properly described the Bauschinger and transient behaviors during unloading. In this work, in order to describe...
the Bauschinger effect and transient behavior under the cyclic loading, two types of combined isotropic-kinematic hardening laws were developed and compared by modifying the non-linear kinematic hardening law previously proposed by Chaboche. In the first law, the anisotropy of the yield function was considered but no particular anisotropic evolution of the back-stress is considered, as commonly done for anisotropic metals; the isotropic back-stress evolution. In the second law, besides the anisotropic yield function, the anisotropic evolution of the back-stress was also implemented considering the directionality of the back-stress evolution: the anisotropic back-stress evolution.

In order to account for the orthotropy of composites, Hill’s 1948 quadratic yield function was utilized. The procedure to separate the isotropic and kinematic hardening curves from the simple tension test data is also discussed along with the stress update scheme based on the incremental deformation theory for the boundary value problem.

CONSTITUTIVE EQUATIONS

Elastic Behavior

Consider uni-directionally-laminated composites in the 1-2 plane under the plane stress condition as
shown in Figure 2. The stress and strain states are described by vectors \( \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T \) and \( \varepsilon = (\varepsilon_{11}^e, \varepsilon_{22}^e, \varepsilon_{12}^e)^T \) for the plane stress state. The stress-strain relationship for orthotropic materials is

\[
\sigma = \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} = \begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{21} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11}^e \\
\varepsilon_{22}^e \\
\varepsilon_{12}^e
\end{bmatrix} = D^e \varepsilon^e
\]  

(1)

where the reduced stiffness \( Q_{ij} \) are

\[
Q_{11} = \frac{E_1}{1-\nu_{12} \nu_{21}}, Q_{12} (= Q_{21}) = \frac{\nu_{12} E_2}{1-\nu_{12} \nu_{21}} = \frac{\nu_{21} E_1}{1-\nu_{12} \nu_{21}}
\]

\[
Q_{22} = \frac{E_2}{1-\nu_{12} \nu_{21}}, Q_{66} = G_{12}
\]  

(2)

where \( E_1, E_2, G_{12}, \nu_{12}, \nu_{21} \) are Young’s moduli in the 1 and 2 directions, shear moduli in the 1-2 plane and Poisson’s ratios, respectively.

**Anisotropic Yield Function**

In order to describe the initial anisotropic yield stress surface, Hill’s 1948 anisotropic yield criterion\(^{10} \) for the plane stress state is considered; i.e.,

\[
\Phi = f(\sigma) - \sigma_{iso}^2 = \sigma^T B \sigma - \sigma_{iso}^2 = 0
\]  

(3)

where \( B = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & b_{66} \end{bmatrix} \) and \( \sigma_{iso}^2 \) is the equivalent stress measuring the size of the yield surface as the first order homogeneous function, while \( f \) is the 2nd order homogeneous function.

**Flow and Evolution Rules**

**Flow Rule with Isotropic Back-Stress Evolution**

The yield surface of the combined type is described by

\[
f(\sigma - a) - \sigma_{iso}^2 = 0
\]  

(4)
where \( \alpha \) is the back-stress by which the current yield surface is translated from an initial position. The plastic work increment becomes

\[
dw = \sigma d\varepsilon^p = (\sigma - \alpha) d\varepsilon^p + \alpha d\varepsilon^p = dw_{iso} + dw_{\alpha}
\]

where \( d\varepsilon^p \) is the plastic strain increment. The plastic work increment becomes

\[
dw = \sigma d\varepsilon^p = \sigma d\varepsilon^p + \alpha d\varepsilon^p = dw_{iso} + dw_{\alpha} \quad (5)
\]

where \( d\varepsilon^p \) is the plastic strain increment. The effective quantities are defined considering the following modified plastic work equivalence relationship; i.e.,

\[
dw_{iso} = \sigma d\varepsilon^p = \sigma d\varepsilon^p + \alpha d\varepsilon^p = dw_{iso} + d\varepsilon^p \quad (6)
\]

where \( d\varepsilon^p \) is the effective plastic strain increment which is conjugated with \( \sigma_{iso} \). In Equation (6), \( \sigma_{iso} \) is obtained by replacing \( \sigma \) with \( \sigma - \alpha \).

As for the effective back-stress increment, \( d\alpha \), the value is obtained from the initial effective stress by replacing \( \sigma \) with \( d\alpha \). The definitions of the effective quantities for the stress, the conjugate plastic strain increment, and the back stress increment are for any anisotropic yield surfaces, which are expressed as first-order homogeneous functions. When plastic deformation and loading are monotonically proportional (from an initial state),

\[
dw_{iso} = (\sigma - \alpha) d\varepsilon^p = \sigma_{iso} d\varepsilon^p
\]

where \( d\varepsilon^p \) is the effective plastic strain increment which is conjugated with \( \sigma_{iso} \). In Equation (6), \( \sigma_{iso} \) is obtained by replacing \( \sigma \) with \( \sigma - \alpha \).

Flow Rule with Anisotropic Back-Stress Evolution

The isotropic back-stress evolution law described in equation (9) and (10) is effective when the kinematic hardening parameters are available for only one direction. However, to account for the directional difference of the back-stress evolution for the highly anisotropic materials such as fiber-reinforced composites, the following anisotropic back-stress evolution law is proposed here, by modifying the isotropic evolution rule:

\[
d\alpha = (d\varepsilon^p - d\varepsilon^p)^T (H d\varepsilon - H d\varepsilon - \alpha) = h_1 d\varepsilon - h_2 d\varepsilon \quad (10)
\]

where \( C \) and \( H \) are fourth order tensors which contain material parameters to be experimentally determined for back-stress evolutions. When the plane stress condition for orthotropic materials is considered, equation (13) becomes

\[
d\alpha = C (\sigma - \alpha) \sigma_{iso} = (\sigma - \alpha) \sigma_{iso} = h_1 d\varepsilon - h_2 d\varepsilon
\]

but

\[
d\alpha = C (\sigma - \alpha) \sigma_{iso} = (\sigma - \alpha) \sigma_{iso} = h_1 d\varepsilon - h_2 d\varepsilon
\]

where \( C \) and \( H \) are fourth order tensors which contain material parameters to be experimentally determined for back-stress evolutions. When the plane stress condition for orthotropic materials is considered, equation (13) becomes

\[
d\alpha = C (\sigma - \alpha) \sigma_{iso} = (\sigma - \alpha) \sigma_{iso} = h_1 d\varepsilon - h_2 d\varepsilon
\]

Note that \( \alpha_i = \alpha_i (\varepsilon) \), while \( \alpha_i = \alpha_i (\varepsilon) \) in general.

Under the simple tension test condition, equations (8)-(10) lead to

\[
d\alpha = h_1 d\varepsilon - h_2 d\varepsilon \quad (11)
\]

If both \( h_1 \) and \( h_2 \) are assumed constant, equation (10) is the 1st order linear differential equation for \( \alpha \), whose solution becomes

\[
\alpha = \frac{h_1}{h_1} (1 - e^{-h_1}) \quad (12)
\]

where, \( \varepsilon = \int d\varepsilon \). After \( \alpha (\varepsilon) \) is measured from the simple tension test involving loading and unloading, \( h_1 \) and \( h_2 \) are obtained by comparing equation (12) with the measured \( \alpha (\varepsilon) \).
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\[
\begin{pmatrix}
\frac{d\alpha_{11}}{d\alpha_{11}} \\
\frac{d\alpha_{22}}{d\alpha_{11}} \\
\frac{d\alpha_{12}}{d\alpha_{11}}
\end{pmatrix} = 
\begin{pmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{pmatrix}
\begin{pmatrix}
n_{11} \\
n_{22} \\
n_{12}
\end{pmatrix}
\begin{pmatrix}
H_{11} & H_{12} & 0 \\
H_{21} & H_{22} & 0 \\
0 & 0 & H_{33}
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} \\
\alpha_{22} \\
\alpha_{12}
\end{pmatrix} 
\begin{pmatrix}
d\alpha_{11} \\
d\alpha_{22} \\
d\alpha_{12}
\end{pmatrix}
\end{equation}
(14)

which can be further simplified assuming \(C_{21} = C_{12} = H_{12} = H_{21} = 0\). Under such circumstances, the following relationships are obtained for monotonously proportional experiments:

\[
\begin{pmatrix}
\alpha_{11} \\
\alpha_{22} \\
\alpha_{12}
\end{pmatrix}^T = 
\begin{pmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{pmatrix}\left(1 - e^{-n_{11}r}\right)\begin{pmatrix}
1 - e^{-n_{22}r} \\
1 - e^{-n_{12}r}
\end{pmatrix}^T
\]
(15)

The six material constants in equation (15) are obtained from the curve fitting of actual stress-strain curves in three deformation modes: simple tension tests in the 1 and 2 directions and the pure shear test.

**STRESS UPDATE SCHEME**

In this paper, the additive decoupling of the total strain increments into the elastic and plastic strain increments, \(d\varepsilon = d\varepsilon^p + d\varepsilon^e\), and the associated flow rule based on the normality rule are assumed. For the numerical formulation, the incremental deformation theory\(^{12}\) was applied to the elasto-plastic formulation based the materially embedded coordinate system. Under this scheme, the strain increments in the flow formulation become discrete true strain increment while the material rotates by the incremental rotation obtained from the polar decomposition at each discrete step.

As for the stress update scheme, the methods developed by Chung\(^{14}\) and Simo and Hughes\(^{15}\) were considered here. In this stress update scheme, the updated stress is initially assumed to be elastic for a given discrete strain increment \(\Delta \varepsilon\). Therefore,

\[
\sigma_{n+1}^T = \sigma_n + D^* \Delta \varepsilon
\]
(16)

where ‘T’ stands for a trial state and the subscript denotes the process time step. Also, preserving the plastic quantities as the previous values,

\[
\bar{\varepsilon}_{n+1}^T = \bar{\varepsilon}_n^T, \alpha_{n+1}^T = \alpha_n
\]
(17)

If the following yield condition is satisfied with the trial values for a prescribed elastic tolerance; i.e.,

\[
f^1\left(\sigma_{n+1}^T - \bar{\sigma}^n\right) - \bar{\sigma}^n(\bar{\varepsilon}_{n+1}^T) < Tol.
\]
(18)

the process time step \(n+1\) is considered elastic.

If the above yield condition is violated, the step is considered elasto-plastic and the trial elastic stress state is taken as an initial value for the solution of the plastic corrector problem. The non-linear equation to solve for \(\Delta \varepsilon\), which enables the resulting stresses to stay on the hardening curves at the new step \(n+1\), is

\[
f^1\left(\sigma_n - \alpha_n + \Delta \sigma - \Delta \alpha_n + \Delta \alpha_n\right) = \bar{\sigma}^n(\bar{\varepsilon}_n + \Delta \varepsilon)
\]
(19)

The equation is applied for each yield surface progressively introduced between the initial and the trial elastic stress state as shown in Figure 3. The predictor-corrector based on the Newton-Raphson method was used to solve equation (19). After obtaining the converged solution of equation (19), the stresses, back stresses and equivalent plastic strain are updated for the next step.

**EXAMPLE SOLUTIONS**

For verification purposes, loading and unloading behaviors were simulated for the uni-axial tension test using the constitutive equation developed. The material parameters are chosen arbitrarily and only isotropic kinematic hardening is considered for simulation. Since the same kinematic hardening parameters are used for the two back-stress evolution rules, the stress-strain curve would be identical. The numerical uni-axial loading-unloading-reverse loading curve identically obtained for both rules is shown in Figure 4. The calculated curve confirms that the constitutive laws clearly show the Bauschinger effect and transient behavior.

Another numerical example utilizes experimental data obtained from the previous work\(^8\) for the analysis.
of stress-strain curves of orthotropic fiber reinforced composites under uni-axial tension and shear tests. The longitudinal (which is usually the fiber direction) and transverse directions coincide with the coordinate system described in Figure 3. The mechanical properties which conform to the characteristics of the unidirectional glass fiber-reinforced composite\textsuperscript{8,16} are:

\begin{align*}
\text{Elastic constants:} \\
E_1 &= 63,000 \text{ (MPa)}, \ E_2 = 9,700 \text{ (MPa)}, \ G_{12} = 7,400 \text{ (MPa)}, \ \nu_{12} = 0.3 \\
\text{Yield function coefficients:} \\
b_{11} &= 6.92\times10^{-5}, \ b_{22} = 1, \ b_{66} = 0.39, \ (b_{12}: \text{no data}) \\
\text{Initial yield stresses:} \\
\sigma_{11y} &= 1800 \text{ (MPa)}, \ \sigma_{22y} = 15 \text{ (MPa)}, \ \sigma_{12y} = 24 \text{ (MPa)}
\end{align*}
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Kinematic hardening equations [8]:
\[ \alpha_{22} = 300(1-e^{-80\varepsilon_{22}}) \text{(MPa)}, \quad \alpha_{12} = 87.5(1-e^{-40\varepsilon_{12}}) \text{(MPa)} \]
(\( \alpha_{11} \): No data)

Fracture stress:
\[ F_{11} = 1800 \text{ (MPa)}, \quad F_{22} = 25 \text{ (MPa)}, \quad F_{12} = 59 \text{ (MPa)} \]

In this example, the kinematic hardening without isotropic hardening is assumed, which means the size of the yield surface is constant during the plastic deformation. Since the yield stress in the longitudinal direction (in the 1-direction) is assumed to be the same as the failure strength, the plastic deformation does not occur in the 1-direction. Therefore, the above anisotropic coefficients for Hill’s 1948 yield function was calculated from the hardening curve in the transverse direction (in the 2-direction).

In Figures 5-6, the measured stress-strain curves are compared with the calculated curves for the uni-axial

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Figure 5. Comparison of the measured stress-strain curves with those calculated using the two evolution rules of kinematic hardening for uni-axial tension and pure shear tests. A: Longitudinal (measured and calculated based on the orthotropic elasticity), B: Transverse (measured and calculated based the anisotropic and isotropic evolution rules), C: Shear (measured and calculated based on the anisotropic evolution rule), D: Shear (calculated based on the isotropic evolution rule)

Figure 6. Comparison of the measured stress-strain curves with those calculated using the two evolution rules of kinematic hardening for uni-axial tension and pure shear tests. A: Longitudinal (measured and calculated based on the orthotropic elasticity), B: Transverse (measured and calculated based the anisotropic evolution rule), C: Shear (measured and calculated based on the anisotropic and isotropic evolution rules), D: Transverse (calculated based on the isotropic evolution rule)
tension tests along the longitudinal and transverse directions (Curves A and B, respectively) and also for the pure shear test (Curve C) in the 1-2 plane. As for the longitudinal uni-axial curves, the measured curves within the elastic range were exactly the same with the calculated ones, which were obtained from the orthotropic elasticity (Curve A). Also, the calculated transverse uni-axial and pure shear curves based the anisotropic evolution rule are exactly the same with the experimental curves (Curves B and C).

The calculated curves based on the proposed anisotropic evolution rule are also compared with the curves calculated based on the isotropic evolution rule in Figures 5-6. In Figure 5, the reference state of the isotropic rule is the transverse uni-axial tension so that the calculated transverse uni-axial curves based on the anisotropic and isotropic rules are the same (Curve B), while the pure shear curve based on the anisotropic curve (Curve C) is quite different from that calculated based on the isotropic rule (Curve D).

In Figure 7, loading and unloading behaviors are compared for those calculated based on the anisotropic and isotropic evolution rules for the transverse uni-axial tension and the pure shear (without considering fractures). In Figure 7(a), the reference state of the isotropic evolution rule is the transverse uni-axial tension so that the transverse uni-axial tension curves are the same for the isotropic and anisotropic rules (Curve A), while the curve for the pure shear based on the anisotropic rule (Curve C) is quite different from that calculated based on the isotropic rule (Curve D).

In Figure 7(b), the reference state of the isotropic evolution rule is the pure shear so that the pure shear curves are the same for the isotropic and anisotropic rules (Curve C), while the curve for transverse uni-axial tension based on the anisotropic rule (Curve A) differs from the transverse uni-axial tension based on the isotropic rule (Curve C). These example solutions shown in Figures 5~7 demonstrate that the kinematic hardening rule with anisotropic back-stress evolution rule is essential to properly describe the constitutive behaviors of highly anisotropic materials such as fiber-reinforced composites.

Figure 7. Comparison of loading and unloading curves calculated based on the two evolution rules of kinematic hardening (a) A: Transverse (based on the anisotropic and isotropic evolution rules), B: Shear (based anisotropic evolution rule), C: Shear (based on the isotropic evolution rule) (b) A: Transverse (based on the anisotropic evolution rule), B: Shear (based on the anisotropic and isotropic evolution rules), C: Transverse (based on the isotropic evolution rule)
CONCLUSIONS

In order to develop the theory of plasticity for orthotropic fiber-reinforced composites, a new combined isotropic-kinematic hardening law with the anisotropic back-stress evolution rule was developed under the elasto-plasticity scheme. To account for the anisotropic yield function, Hill's quadratic yield function was used. The stress update scheme based on the incremental deformation theory was also developed. Example solutions confirmed that the new constitutive equation well represented the Bauschinger and transient behaviors, as well as the directional dependence of the back-stress evolution for the highly anisotropic materials such as fiber-reinforced composites.

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